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How do help "Multigrid Principles based on Numerical Solutions of Partial Differential Equations" for Smoothing Process (Concepts)?

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Abstract

In Twenty century, Partial Differential Equations (PDE) are solved by Numerical Approach such as Adam – Smith Methods, Numerical Simulation Methods, Finite Difference methods etc., In this article we discuss about difficulties of solve either Differential equations or PDE in 3 – Dimensional cases. So that we discuss only 1 – Dimensional Multigrid Methods (M. M) and we discuss about M. M. Errors, Corrections, Type of grid such as Coarse Grid (C. G) and Fine Grid (F. G), Smoothing, non – smoothing approximation of C. G. Finally, we explain that M. G. M. works by decomposing problem into separate length scale and also using an iterative method. This method optimizes errors deduction in the length scales globally. In Multigrid Methods (MgM) several sub – routines must be developed to pass the data from C. G to F. G (Interpolation) from F. G to C. G (Reduction) and correction of the error at each grid interval (Smoothing), simply we have results reaction as (Reduction)C.G⇄F.G (Interpolation).

Introduction

We know that Partial Differential Equations (PDE) can be solved by analytically and numerically. Most of the PDE can be analyzed with numerical approach, for example Elliptic and Hyperbolic PDE solved by Gal grin's Method and other methods also (Hemker, 1990). In this article we develop the application of Multi – Grid Methods (MgM) with the support of Functional Analysis concepts (Hackbusch, 1977). The (MgM) was firstly, introduced by Hack Busch (1977) who used reliable methods. Fredrickson (1974) developed a powerful (Mg) Algorithm for the Poisson Equation.

We will discuss here the real merits of (MgM). By the fundamental concepts of (MgM) which is introduced by Briggs (1987). For this concept, we can choose One – Dimensional Problem such as

$u'' = -f(x)$ in open interval $(0,1) = \gamma$ (say) (1)

And also, assume that,

$u'(0) = u'(1)$ with respect to x (2)

Define a **Computational grid** denoted by G_C and

$$G_C = \left\{x \in \mathbb{R} | x = x_j = jh, for\ j = 1, 2, \dots, 2n; h = \frac{1}{2n}\right\} \quad (3)$$

Note that, the set $\{x_j\}$ are called the **Vertices** of the G_C.

Since, the equation (1) can be described with finite differences concepts as

$$\begin{cases} \frac{1}{h^2}(2u_1 - u_2) = f_1 \\ \frac{1}{h^2}(-u_{j-1} + 2u_j - u_{j+1}) = f_j \\ \frac{1}{h^2}(u_{2n-1} + u_{2n}) = \frac{1}{2}f_{2n} \end{cases} \quad (4)$$

$Where, f_j = fx_j\ and\ \boldsymbol{u}_j \approx u(x_j)$

For equation (1), the solution is denoted by $\boldsymbol{u}$ and the solution of equation (3) by $\boldsymbol{u}$ and its value of $\boldsymbol{u}$ in x_j by $\boldsymbol{u}_j$ and then we conclude that $\boldsymbol{u}_j$ is near to the solution in the grid vertex x_j. That is why, equation (3) is called a **Vertex Centered Discretization** (VCD)

And also assume that without loss of generality in the grid G_C the number of members is even. We can write the equation (3) in the Matrix - Vector Form such as

$$A\boldsymbol{u} = \boldsymbol{f} \quad (5)$$

$Where$, the coefficient matrix A, it is large, sparse and non – zero matrix.

Now, Equation (4) can be solved by **Gauss – Seidal Iteration Method**.

Let us take, the initial stage as u^0 of equation (4) and it can be given by

$$\begin{cases} 2u_1^m = u_2^{m-1}h^2 f_1 \\ -u_{j-1}^m + 2u_j^m = u_{j+1}^{m-1} + h^2 f_j; j = 2, 3, \dots, 2n-1 \\ -2u_{2n-1}^m + u_{2n}^m = \frac{1}{2}h^2 f_{2n} \end{cases} \quad (6)$$

To identify the convergence of the solution, we will use **Fourier Concepts**. In this situation, we take periodic boundary conditions such as

$u(0) = u(1)$ (7)

This above type of periodic grid function (7), which also be mentioned by the **Fourier Series** form such as

$$e_j^m = \sum_{a=0}^{2n-1} C_a^m \left(e^{ij\theta_a}\right); for\ \theta_a = \frac{\pi a}{n} \quad (8)$$

Now, we can find the error that is,

$e^m = u^m - u^\infty$, it is also periodic and satisfies

$$-e_{j-1}^m + 2e_j^m = e_{j+1}^{m-1}; for\ e_j^m = e_{j+2n}^{2m} \quad (8)$$

Above type of periodic grid function (8), which also be mentioned by the Fourier Series form such as

$$e_j^m = \sum_{a=0}^{2n-1} C_a^m \left\{e^{ij\theta_a}\right\}; for\ \theta_a = \frac{\pi a}{n} \quad (9)$$

Due to the Orthogonality of $\left\{e^{ij\theta_a}\right\}$, simply we can take

$e_j^{m-1} = C_a^{m-1}\{e^{ij\theta_a}\}$ in equation (8), finally we get

$$C_a^m = g(\theta_a)C_a^{m-1}\ for\ g(\theta_a) = \frac{e^{i\theta_a}}{2-e^{-i\theta_a}} \quad (10)$$

Note that, in equation (10), the function $g(\theta_a)$ it is called the **Amplification Factor** (A. F.). This A. F. will help to measures Growth or Decay of Fourier Mode of the error during an iteration (Kettler, 1982). By some useful simple calculations, we will reach **Fourier Mode Error** (F. M. E.) as

$$\left\{|g(\theta_a)| \text{ such that}, \theta_a = \frac{\pi a}{n}; for\ a = 0, 1, 2, \dots, 2n-1\ \ and\ |g(0)| = 1\right\} \quad (11)$$

Again, note that, by the periodic boundary condition of the solution of equation (11) is obtained **Purely Constant** (Adams, 1989). Therefore, we simply ignore Fourier Mode as zero without loss of generality, decays during iterations

$\Longrightarrow$Equation (11) is not correct measure of convergence, we can find the value as

$$\max\left\{|g(\theta_a)| \text{ such that}, \theta_a = \frac{\pi a}{n}; for\ a = 0, 1, 2, \dots, 2n-1\right\} = |g(\theta_1)| = \sqrt{\{1 + 2\theta_1^2 + O(\theta_1^4)\}}$$

$$\Longrightarrow \max\left\{|g(\theta_a)| \text{ such that}, \theta_a = \frac{\pi a}{n}; for\ a = 0, 1, 2, \dots, 2n-1\right\} = 1 - 4\pi^2h^2 + O(h^4) \quad (12)$$

The rate of convergence as depend upon $h \longrightarrow 0$

Sometime, it is true for all elliptic equations except special cases, so that, it is called **Basic Iterations method** (BIM)

The Important Needful of Multigrid Concepts

BIM of convergence will be improved with Multigrid Methods.

Since $|g(\theta_a)|$ decreasing ($\downarrow$) as a increasing ($\uparrow$), so that, we got the following results

a. For Long Wave Length, F. M of a near to 1, that is, delay slowly and then we get $|g(\theta_a)| = 1 - O(h^2)$
b. For Short Wave Length F. M of a reduced rapidly.

From those above two results, the importance of Multigrid Principle is to be approximate the smooth for the result (a), part of the error on Courser Grid (Bank & Sherman, 1981). The non – smooth or rough part is reduced with small number free from the value of h of iteration with a BIM on the fine Grid.

For further Multigrid procedure, we need some useful definitions

Rough Wave Number Set

It is denoted by R_w and it is defined by

$$R_w = \left\{\theta_a = \frac{\pi a}{n};, a \geq cn, for\ a = 0, 1, 2, \dots, 2n-1\right\} \quad (13)$$

$where\ 0 < c < 1$, it is a fixed constant independent of n

The Smoothing Factor

It is denoted by S_f and it is defined by

$$S_f = \max\{|g(\theta_a)|: \theta_a \in R_w\} \quad (14)$$

In Basic Iterative Method (B. I. M) $S_f < 1$, that is, it is bounded away from 1, and uniformly in h, so that, this method is called **Smoother** and also S_f depends on Iterative method and our problem. For Gauss – Seidel Iteration and present our model the value of S_f it is easily calculated. Due to equation (10), $|g(\theta_a)|$ decreases monotonically and also, we get after few calculations

$$S_f = \frac{1}{\sqrt{(5-4\cos(c\pi))}} \quad (15)$$

Therefore, **Gauss – Seidal Iteration Model is a Smoother**. This model is suitable, for the selection of constant suitable c.

Secondly, the **Fourier Mode** that cannot be represented on **the Coarse Grid**, which needs to reduce by B. I. M. That is why, for Coarse Grid, we simply doubling the mesh - size h of G_c. That is, due to equation (8), simply change $2n \ by\ n, but\ n\ is\ even$, for our convince. Then, the balanced wave numbers are defined as **Non – Smooth** and also, we got by using equation (13) with constant

$$c = 1 \quad (16)$$

Using $c = 1$, the equation (15) gives,

$$S_f = \sqrt{\frac{1}{\sqrt{(5-4\cos(c\pi))}}} = \frac{1}{\sqrt{(5)}} \quad (17)$$

Equation (17) is the **Smoothing Factor** for Gauss – Seidal Iteration. These concepts of Gauss – Seidal Iteration, first introduced by Brandt (1977).

Suppose that, the given Smoothing Factor S_f, it is not periodic and then S_f are qualitatively correct but, only not correct for the case of **Singular Perturbation Model**.

Thirdly, with variable coefficients, a smoother factor S_f acting very well, that is, the S_f performs very well in the "**Frozen Coefficient**" case and also act very well for variable coefficients also. That is, in the frozen coefficient case the constant c, as a set of constant coefficients with coefficients same as the values of **the variable coefficients** under enough large sample points in the domain $(0, 1)$

Two Grid Procedure

For this case, the smoothing part of the error can be reduced by the C. Gs, it is enough to learn the Two Grid method for our model problem.

Before create this C. G. Model Approximation, we need the following definitions, that is,

Vertex – Centered Coarsening (Definition)

Consider a Course Grid denoted by $\overline{G}_c$ and defined by

$$\overline{G}_c = \left\{x \in \mathbb{R}: x = x_j = j\overline{h}, for\ j = 1, 2, \dots n\ and\ \overline{h} = \frac{1}{n}\right\} \quad (19)$$

Note that, all vertices of $\overline{G}_c \in G_c$, that is why, it is called **Vertex – Centered Coarsening**. And also, the grid G it is called **Fine Grid** (F. G.)

Consider the functions

$M: G_c \longrightarrow \mathbb{R}\ \ and\ \overline{M}: \overline{G}_c \longrightarrow \mathbb{R}$, these two functions are **the set of Fine and Course grid functions** respectively.

Definitions for Two Operators

Now we will introduce two operators, they are **Prolongation Operator** denoted by $\boldsymbol{P}: \overline{M} \longrightarrow \mathbb{R}$,

It is defined by Linear Interpolation that is,

$$\begin{cases} P\overline{u}_{2j} = \overline{u}_j \\ P\overline{u}_{2j+1} = \frac{1}{2}\left(\overline{u}_j + \overline{u}_{j+1}\right) \end{cases} \quad (20)$$

All vertices, $\overline{u}_j, for\ j = 1, 2, \dots, n+1$, they are Course Grid Quantities.

Another one operator is **Restriction Operator** denoted by R as $R: U \longrightarrow \overline{U}$ is defined by

$$Ru_j = \frac{1}{4}\left\{u_{2j-1} + 2u_{2j} + u_{2j+1}\right\} \quad (21)$$

$Where, u_j$ it is defined **Zero,** Outside G_c that is, outside of Find Grid Set.

Form these two operators $P\ and\ R$, that is, the equations (20) and (21) the prolongation and Restriction Operators matrices are $P\ and\ R$ have very good relation such as

$\boldsymbol{R} = \frac{1}{2}P^{Transpose}$ simply $\boldsymbol{R} = \frac{1}{2}\boldsymbol{P}^T$

Now we develop the fine grid equation, in the Matrix - Vector Form

$$A\boldsymbol{u} = \boldsymbol{f} \quad (5)$$

It is also approximated by a Course – Grid Matrix – Vector Form equation such as

$$\overline{A}\overline{\boldsymbol{u}} = \overline{\boldsymbol{f}} \quad (22)$$

The coefficient matrix $\overline{A}$ is obtained by Discretizing Equation (1), so that, it is called **Discretization Course Grid Approximation**.

Now, we can develop the Fine Gerid Problem for equation (5) by using **Inner Product Approach**.

Consider,

$$\langle A\boldsymbol{u}, \boldsymbol{u} \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, for\ \ \boldsymbol{u} \in U\ and\ for\ every\ \boldsymbol{v} \in V \quad (23)$$

With $\langle ,.,\rangle$ the standard inner product on U

It is equivalent the fine grid problem in Course Grid Approximation, that is,

Like Fine Grid matrix A, we can find the matrix $\overline{A}$, it is Coarse Grid matrix, and it is obtained from the equation,

$u'' = -f(x)$ in open interval $(0, 1) = \gamma$ (say) (1)

The matrix $\overline{A}$ is called **Discretization Course Grid Approximation**. The alternative Inner product approach, since the fine grid problem

$$A\boldsymbol{u} = \boldsymbol{f}$$

It is approximated by course Grid Approximation problem

$$\overline{A}\overline{\boldsymbol{u}} = \overline{\boldsymbol{f}}$$

Aim:

To find the approximate solution $\mathbf{P}\overline{\boldsymbol{u}}\ \ with\ \overline{\boldsymbol{u}} \in \overline{U}$

Let us take restriction of the test function $\boldsymbol{v}$ to a subspace with equal dimension as $\overline{\boldsymbol{u}}$, that is the functions of the type $\overline{\boldsymbol{P}}\overline{\boldsymbol{v}}\ with\ \overline{\boldsymbol{v}} \in \overline{U}$ and $\overline{\boldsymbol{P}}$**,** it is also a prolongation operator but it differs from **P.** Now we get

$$\langle A\boldsymbol{P}\overline{\boldsymbol{u}}, \overline{\boldsymbol{P}}\overline{\boldsymbol{v}} \rangle = \langle \boldsymbol{f}, \overline{\boldsymbol{P}}\overline{\boldsymbol{v}} \rangle,\ for\ \overline{\boldsymbol{u}} \in \overline{U}\ and\ for\ every\ \overline{\boldsymbol{v}} \in \overline{U} \tag{24}$$

Equivalent to, otherwise we have

$$\langle \overline{\boldsymbol{P}}^{*}\boldsymbol{A}\boldsymbol{P}\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}} \rangle = \langle \overline{\boldsymbol{P}}^{*}\boldsymbol{f}, \overline{\boldsymbol{v}} \rangle\ \ for\ \overline{\boldsymbol{u}} \in \overline{U}\ and\ for\ every\ \overline{\boldsymbol{v}} \in \overline{U} \tag{25}$$

Note that $\overline{\boldsymbol{P}}^{*}$**,** it is either adjoint of **P** or transpose of **P**

We know that, the inner product $\langle ,.,\rangle$ it is over $\overline{U}$

From equation (25) we conclude that

$$\overline{A}\overline{\boldsymbol{u}} = \overline{\boldsymbol{f}} \tag{26}$$

With $\overline{\boldsymbol{A}} = \boldsymbol{RAP}$ (27)

And also, we have, $\overline{\boldsymbol{f}} = \boldsymbol{Rf}\ \ where\ \ \boldsymbol{R} = \overline{\boldsymbol{P}}^{*}$

Results and Discussion

Remark:

The matrix $\overline{\boldsymbol{A}}$, it is called **Galerkin Course Grid Approximations**.

Result – 1: Galerkin Matrix $\overline{\boldsymbol{A}}$ it is equivalent to Discretization Course Grid Approximation

By using matrices $\boldsymbol{A}$, $\boldsymbol{P}, and\ \boldsymbol{R}$ for the matrix $\overline{\boldsymbol{A}}$ we will get the following results

$$\begin{cases} \overline{A}\overline{u}_1 = \frac{1}{(\overline{h})^2}(2\overline{u}_1 - \overline{u}_2) \\ \overline{A}\overline{u}_j = \frac{1}{(\overline{h})^2}\left(-\overline{u}_{j-1} + 2\overline{u}_j - \overline{u}_{j+1}\right), for\ j = 1, 2, \dots, n-1 \\ \overline{A}\overline{u}_n = \frac{1}{(\overline{h})^2}(-\overline{u}_{n-1} + \overline{u}_n) \end{cases} \tag{28}$$

Note that, The Galerkin matrix $\overline{\boldsymbol{A}}$ it is equivalent to the left – hand side of discretized with Finite – Difference, so that we conclude that

Galerkin Matrix $\overline{\boldsymbol{A}} \approx$ Discretization Course Grid Approximation

Result – 2: To Calculate Course Grid Correction

Let us take, $\widetilde{u}$, it is an approximation, solution of $A\boldsymbol{u} = \boldsymbol{f}$

Therefore, the error denoted by e , it occurs the difference between $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{u}$

$$\Longrightarrow \widetilde{\boldsymbol{u}} \sim \boldsymbol{u} = \boldsymbol{e} \tag{29}$$

Equation (29) is approximated on Course Grid and therefore we have

$$\boldsymbol{Ae} = -\boldsymbol{r} = \boldsymbol{A}\widetilde{\boldsymbol{u}} - \boldsymbol{f}$$

Therefore, the Course Grid Approximation $\widetilde{\boldsymbol{u}}$ of $-\boldsymbol{e}$, satisfies

$$\overline{A}\overline{\boldsymbol{u}} = \boldsymbol{Rr}$$

And also, it will be solved exactly by the Two – Grid – Method.

Now the Course Grid Correction to be added to $\widetilde{\boldsymbol{u}}$ and it is $\boldsymbol{P}\overline{\boldsymbol{u}}$

$\Longrightarrow \widetilde{\boldsymbol{u}}$ is same as $\widetilde{\boldsymbol{u}} + \boldsymbol{P}\overline{\boldsymbol{u}}$

To Develop Two – Grid Algorithm

First, we assume that the initial stage is $\boldsymbol{u}_1^0$

Then the first iteration starts such as

$$\left|\begin{array}{c} \boldsymbol{u}^{\frac{1}{3}} = S(\boldsymbol{u}^0, \quad \boldsymbol{A}, \quad \boldsymbol{f}, \quad v_1) \\ Let\ us\ take, \boldsymbol{r}\ as \\ \boldsymbol{r} = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}^{\frac{1}{3}}, since \\ \overline{\boldsymbol{u}} = \frac{\boldsymbol{1}}{\boldsymbol{A}}\boldsymbol{r},\ and\ also \\ \boldsymbol{u}^{\frac{2}{3}} = \boldsymbol{u}^{\frac{1}{3}} + P\overline{\boldsymbol{u}},\ and\ then \\ \boldsymbol{u}^1 = S\left(\boldsymbol{u}^{\frac{2}{3}}, \quad \boldsymbol{A}, \quad \boldsymbol{f}, \quad v_2\right) \\ \boldsymbol{u}^1 = S\left(\boldsymbol{u}^{\frac{2}{3}}, \quad \boldsymbol{A}, \quad \boldsymbol{f}, \quad v_2\right) \\ If\ \boldsymbol{u}^0 = \boldsymbol{u}^1 \\ Stop, Iterations, otherwise, Continue\ iterations \text{ until we will reach} \\ \boldsymbol{u}^0 = \boldsymbol{u}^1, for\ i = 2, 3, \dots, \end{array}\right| \quad (30)$$

That is, the initial stage is equal to first iteration and then stop iterations, otherwise continue iterations until we will reach

$\boldsymbol{u}^0 = \boldsymbol{u}^i, for\ i = 2, 3, \dots$

And note that, the number of two – grid iterations $(ntg.)$ carried out stands for v_1, this results as

$S(\boldsymbol{u}^0, \quad \boldsymbol{A}, \quad \boldsymbol{f}, \quad v_1)\ as\ v_1$, smoothing iterations.

For example, take the previous Gauss – Seidal – Method, and then apply to

$\boldsymbol{Au} = \boldsymbol{f}$

Start with initial stage as $\boldsymbol{u}^0$, and then the first application of S, it is called **Pre- Smoothing** and also the second application is called **Post – Smoothing**.

Analyzing Two – Grid Concepts

1. Clearly, the rate of Converges of Two – Grid Methods (Multi – Grid Method) is free from the mesh – size that is, h
2. We can analyze for $v_1 = 0$ (No – Pre – Smooth), by using.
3. We will calculate, Coarse Grid Correction

Since, the error occurs by using (30) [Two – Grid – Algorithm], we get,

$\boldsymbol{e}^{\frac{2}{3}} = \boldsymbol{u}^{\frac{2}{3}} - \boldsymbol{u}$, and it satisfies.

$$\boldsymbol{e}^{\frac{2}{3}} = \boldsymbol{e}^{\frac{1}{3}} + \boldsymbol{P}.\overline{\boldsymbol{u}}^{\frac{1}{3}} = \boldsymbol{E}\boldsymbol{e}^{\frac{1}{3}} \quad (31)$$

Now, we define Iteration Matrix or Error Amplifications Matrix $\boldsymbol{E}$ as

$$\boldsymbol{E} = \boldsymbol{1} - \boldsymbol{P}\left(\frac{\boldsymbol{1}}{\boldsymbol{A}}\right)\boldsymbol{RA} \quad (32)$$

Again, we will express $\boldsymbol{e}^{\frac{2}{3}}$ interms of $\boldsymbol{e}^{\frac{1}{3}}$, but this is possible for only one 0 dimensional cases.

Let us take,

$$\boldsymbol{e}^{\frac{1}{3}} = \boldsymbol{d} + \boldsymbol{P}\overline{\boldsymbol{e}} \ with \ \overline{\boldsymbol{e}}_j = \boldsymbol{e}_{2j}^{\frac{1}{3}}$$

$$\Longrightarrow \boldsymbol{e}^{\frac{2}{3}} = \boldsymbol{E}\boldsymbol{e}^{\frac{1}{3}} = \boldsymbol{E}\boldsymbol{d} \tag{33}$$

Since, our assumption, that is, $\boldsymbol{e}^{\frac{1}{3}} = \boldsymbol{d} + \boldsymbol{P}\overline{\boldsymbol{e}} \ with \ \overline{\boldsymbol{e}}_j = \boldsymbol{e}_{2j}^{\frac{1}{3}}$, we get

$$\begin{cases} \boldsymbol{d}_{2j} = 0 \\ \boldsymbol{d}_{2j+1} = -\frac{1}{2}\boldsymbol{e}_{2j}^{\frac{1}{3}} + \boldsymbol{e}_{2j+1}^{\frac{1}{3}} - \frac{1}{2}\boldsymbol{e}_{2j+2}^{\frac{1}{3}} \end{cases} \tag{34}$$

And since, $\boldsymbol{RAd} = \boldsymbol{0}$

$$\Longrightarrow \boldsymbol{e}^{\frac{2}{3}} = \boldsymbol{d} \tag{35}$$

Verifying Smoothing

Consider the second application of S , that is, post-Smoothing by Gauss – Sedal – Iterations (with initial stage as $\boldsymbol{u}^0$), in this case, we get the error after post – smoothing as

$\boldsymbol{e}^1 = \boldsymbol{u}^1 - \boldsymbol{u}$, and it is related to $\boldsymbol{e}^{\frac{2}{3}}$, by using finite difference concept, we get

$$\begin{cases} 2\boldsymbol{e}_1^1 = \boldsymbol{e}^{\frac{2}{3}} \\ -\boldsymbol{e}_{j-1}^{\frac{2}{3}} + 2\boldsymbol{e}_j^1 = \boldsymbol{e}_{j+1}^{\frac{2}{3}}, j = 2, 3, \dots, 2n-1 \\ -\boldsymbol{e}_{2n-1}^1 + \boldsymbol{e}_{2n}^1 = 0 \end{cases} \tag{36}$$

The difference between by using (34) and (35), we rewrite equation (36) as

$$\begin{cases} \boldsymbol{e}_1^1 = 0 \\ \boldsymbol{e}_{2j+1}^1 = \frac{1}{2}\boldsymbol{e}_{2j}^1, j = 1, 2, \dots, n-1 \\ \boldsymbol{e}_{2n}^1 = \boldsymbol{e}_{2n-1}^1 \end{cases} \tag{37}$$

By using induction, we can easily verify that, in the norm relations that is,

$$\left\|\boldsymbol{e}_{2j}^1\right\| \leq \frac{2}{3}\|\boldsymbol{d}\|_\infty = \max\left\{\left|\boldsymbol{d}_j\right|, j = 1, 2, \dots, 2n\right\} \tag{38}$$

And Since, $\boldsymbol{d} = \boldsymbol{e}^{\frac{2}{3}}$,

Gauss – Sedal – method reduces the maximum norm of the error by a factor $\frac{2}{3}$ or less

That is, we conclude that, and find that, the rate of convergence, by using

$$\left\|\boldsymbol{e}^{\boldsymbol{1}}\right\|_\infty \leq \frac{2}{3}\left\|\boldsymbol{e}^{\boldsymbol{1}}\right\|_\infty \tag{39}$$

By equation (39), we again conclude that,

The rate of convergence is independent of the mesh size h

This is the most useful **property of Multigrid Method**

Conclusion

Final Touch in our article:

By using,

$\boldsymbol{RAd} = \mathbf{0}$ and

$\boldsymbol{e^{\frac{2}{3}} = d}$

We can easily show that,

$$\boldsymbol{RAe^{\frac{2}{3}} = 0} \qquad (40)$$

Therefore, the applications of equation (40), gives us the result as, it takes local weighted average with positive weights, from this we conclude final touch from the equation (40) $\boldsymbol{Ae^{\frac{2}{3}}}$ has many sign changes such as (either positive or negative simultaneously). For these reasons this is rough.

Again, since, $\boldsymbol{Ae^{\frac{2}{3}} = Au^{\frac{2}{3}} - f,}$

This is residual and this residual also rough, that is why, the smoother is more significant essential principle than reducing this Two – Grid method for non – smooth residual concepts.

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